ON THE DENSITY OF WEAK POLIGNAC NUMBERS

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Abstract. Let k be an integer which is the difference between prime numbers infinitely often. It is known that there are infinitely many such k and, in this paper, we give a new unconditional proof that these k have positive density and improve on current bounds, assuming a strong hypothesis.

In 2013, Yitang Zhang proved the bounded gap conjecture [5] which asserted that there exists some natural number k such that there exist infinitely many pairs of primes whose difference is precisely k. We call any k that satisfies such a property a *weak Polignac number*.

More precisely, we first ask what sorts of finite sets $\mathcal{H}$ we could construct such that their translates $n + \mathcal{H}$ could be prime infinitely often. That is to say, we want to create a set $\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$ where, for all primes p, there is some natural number m such that

$$h_i \not\equiv m \pmod p$$

for all $1 \le i \le k$. Any such $\mathcal{H}$ is called an *admissible set*.

This theorem of Zhang's (which was later improved upon by James Maynard [1] and Polymath Project 8 [3] does this by saying that the translates $n + \mathcal{H}$ of an admissible set $\mathcal{H}$ contain at least two primes infinitely often. In other words

Theorem 1 (Maynard–Polymath–Zhang, 2014)**.** *Suppose that* $\mathcal{H} = \{h_1, h_2, \ldots, h_k\}$ *is an admissible set and define the difference set*

$$\mathcal{D} = \{h_j - h_i : h_i < h_j\}.$$

Then, for $k = 50$*,* $\mathcal{D}$ *contains a weak Polignac number.*

If we assume stronger conditions such as the Elliott–Halberstam conjecture or the generalised Elliott–Halberstam conjecture then we get the above theorem with $k = 5$ and $k = 3$ respectively.

If we let $\mathcal{P}(x)$ be the number of weak Polignac numbers less than or equal to x then Pintz has shown [2] that

$$\liminf_{x\to\infty} \frac{\mathcal{P}(x)}{x} > 0$$

and, in fact, he obtained a positive bound for the lower density. This paper gives an unconditional bound which, while not being as strong as Pintz', has a slightly simpler proof so that, if all we care about is that the lower density is positive, this method will suffice. We also

give a bound which improves upon Pintz' under the assumption of the generalised Elliott–Halberstam conjecture and show that, if it is not optimal, then the methods used can improve upon the bound by no more than 1/36.

The method used here is a packing problem: we construct an infinite sequence of admissible sets $\mathcal{H}^n$ such that the difference sets $\mathcal{D}^n$ are disjoint. Previous work in the field has looked for admissible sets of minimal diameter. These are very useful for lowering the bound on the Maynard–Polymath–Zhang theorem but are not very suited to this packing problem as they are rather irregular.

To that end we define a *regular admissible set of size* k to be an admissible set of the form

$$\mathcal{H}_k^n = \{0, nP(k), \ldots, (k-1)nP(k)\}$$

where $P(k) = \prod_{p \leq k} p$ is the k^{th} primorial and note that their difference sets are

$$\mathcal{D}_k^n = \{nP(k), \ldots, (k-1)nP(k)\}.$$

Consider all regular admissible sets of size k $(\mathcal{H}_k^n)_{n\in\mathbb{N}}$. We want to find a subset $\mathcal{N} \subseteq \mathbb{N}$ such that the sequence $(\mathcal{D}^n)_{n\in\mathcal{N}}$ is pairwise disjoint and which pack into the interval $[1, x]$.

We first note that there can be at most $\frac{x}{(k-1)P(k)}$ different regular admissible sets which have difference sets which are entirely contained within $[1, x]$. Then, for $n < m$, $\mathcal{D}_k^n$ overlaps with all regular admissible sets $\mathcal{D}_k^m$ such that

$$im = jn$$

whenever $1 \leq i < j \leq k-1$. There are $\frac{1}{2}(k-1)(k-2)$ different choices of i and j so every regular admissible set included in our collection excludes no more than that many other potential admissible sets. Therefore

$$\begin{aligned}
\mathcal{P}(x) &\geq \left[\frac{x}{(k-1)P(k)}\right] - \frac{(k-1)(k-2)}{2}\sum_{i=1}^{x} 1_{\mathcal{N}}(i) \\
&\geq \left[\frac{x}{(k-1)P(k)}\right] - \frac{(k-1)(k-2)}{2}\mathcal{P}(x)
\end{aligned}$$

which implies that.

$$\mathcal{P}(x) \geq \frac{2x}{(k-1)((k-1)(k-2)+2)P(k)} + O(1).$$

Dividing through by x gives the following result.

Theorem 2. *Suppose that all admissible sets of size k contain a weak Polignac number in their difference sets. Then*

$$\liminf_{x\to\infty} \frac{\mathcal{P}(x)}{x} \geq \frac{2}{(k-1)((k-1)(k-2)+2)P(k)}.$$

Putting $k = 50$ into this formula shows that, unconditionally, the weak Polignac numbers have density greater than $\frac{1}{35{,}462{,}538{,}431{,}226{,}065{,}088{,}930} > 2.819 \times 10^{-23}$. Then we get a new derivation of a theorem of Pintz [2]

Corollary 3. *The lower asymptotic density of the weak Polignac numbers is positive.*

Which, in turn, allows us to apply such wonderful theorems as that of Szemerédi's [4] which tells us that there are arbitrarily long arithmetic progressions within the prime numbers.

Furthermore, under Elliott–Halberstam ($k = 5$) the density is greater than $\frac{1}{840} > 0.00119$ and, under Generalised Elliott–Halberstam ($k = 3$), $\frac{1}{24} = 0.041\dot{6}$.

Actually, under the Generalised Elliott–Halberstam conjecture we can do a little better. Let

$$\mathcal{A} = \{6n \leq x - 2 : n \in \mathbb{N}\}$$

and write $\mathcal{A} = \{a_1, a_2, \ldots, a_N\}$ where $a_i = 0$ whenever $3 \mid i$ and $a_i \neq 0$ otherwise. We can also insist that the non-zero a_i are strictly decreasing and define

$$\mathcal{H}^n = \{0, 2n, 2n + a_n\},$$

where $3 \nmid n$ and $n \leq [x/6]$.

These sets are admissible (as all of their elements are congruent to 0 modulo 2 and either 0 or 2 modulo 3). Also all the elements of the difference sets are contained within $[2, x]$ (as the largest element is $2 + a_1 \leq x$ by definition). Therefore, in this case,

$$\liminf_{x\to\infty} \frac{\mathcal{P}(x)}{x} \geq \frac{1}{6}.$$

We might also like to ask just how successful this stratagem can be. We know that the k-element admissible sets with the smallest difference sets are regular admissible sets with $k - 1$ elements so we can pack no more than

$$\left[\frac{x}{2(k-1)}\right]$$

even difference sets into $[1, x]$. This gives a maximum packing density of $\frac{1}{2(k-1)}$ assuming only the bounded gaps conjecture holds for k and not for $k - 1$.

We can improve upon this in particular cases, however. Take $k = 3$ and note that the maximum cardinality for the difference set of an admissible set is 3 and the minimum is 2. Moreover, this minimum is achieved precisely when the admissible set is regular. We know that regular admissible sets can only be made up of multiples of $P(k) = 6$ and they will have $k - 1 = 2$ elements so there are no more than

$$\left[\frac{x}{12}\right] \sim \frac{x}{12}$$

even regular admissible sets with disjoint difference sets contained in $[1,x]$. We can therefore have an extra

$$\left[\frac{\left[\frac{x}{2}\right]-2\left[\frac{x}{12}\right]}{3}\right]\sim\frac{1}{9}$$

irregular admissible sets. This tells us that we can have no more than

$$\frac{x}{12}+\frac{x}{9}=\frac{7x}{36}$$

admissible sets whose difference sets are disjoint and contained in $[1,x]$ for large enough x. Therefore, only assuming the generalised Elliott–Halberstam conjecture, the maximum packing density is $7/36$.

References

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